T.D.C. Part. II
Paper: III

Jacobians:

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Prestudy:

At first, we have need to study about more than one dependent variables depending upon more than one independent variables and their partial derivatives i.e. partial derivatives of dependent variables with respect to independent variables. We have also need to study about determinant and its properties.

Definition:- Let $u_1, u_2, u_3, \ldots, u_n$ are dependent variables depending upon independent variables $x_1, x_2, \ldots, x_n$ s.t.

$$u_1 = u_1(x_1, x_2, \ldots, x_n)$$
$$u_2 = u_2(x_1, x_2, \ldots, x_n)$$
$$\vdots$$
$$u_i = u_i(x_1, x_2, \ldots, x_n)$$
$$\vdots$$
$$u_n = u_n(x_1, x_2, \ldots, x_n).$$

and $u_1, u_2, \ldots, u_n$ be n differentiable functions of n variables $x_1, x_2, \ldots, x_n$, then the Functional Determinant of the functions $u_1, u_2, \ldots, u_n$ w.r.t. $x_1, x_2, \ldots, x_n$ is called the Jacobian, denoted by $J\left(\frac{u_1, u_2, \ldots, u_n}{x_1, x_2, \ldots x_n}\right)$ or

J simply or $\dfrac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)}$ and defined as $J = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_i} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_i} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \vdots & & & & & \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_i} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$

where $1 \le i \le n$. i.e. $i = 1, 2, \ldots, n$.

It is also denoted by $J(u_1, u_2, \ldots, u_n)$ sometimes.

Now, let us discuss some examples as followings:

Example No.: 1 If $x = r\cos\theta$, $y = r\sin\theta$, then find $\frac{\partial(x,y)}{\partial(r,\theta)}$.

Solution:– We have $x = r\cos\theta$ — (i) & $y = r\sin\theta$ — (ii)

Differentiating (i) and (ii) partially w.r.t. r, we have

$\frac{\partial x}{\partial r} = \cos\theta$ & $\frac{\partial y}{\partial r} = \sin\theta$ ——— (iii)

Again partially differentiating (i) and (ii) w.r.t. θ, we have

$\frac{\partial x}{\partial \theta} = -r\sin\theta$ & $\frac{\partial y}{\partial \theta} = r\cos\theta$ ——— (iv) $\left(\text{Since } \frac{\partial \cos\theta}{\partial\theta} = -\sin\theta,\ \frac{\partial \sin\theta}{\partial\theta} = \cos\theta\right)$

We know that

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \quad (\text{Using (iii) \& (iv)})$$

$$= \cos\theta \cdot r\cos\theta - \sin\theta \cdot (-r\sin\theta) = r\cos^2\theta + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta) = r$$

Hence $\frac{\partial(x,y)}{\partial(r,\theta)} = r$.

Example No.: 2 If $u_1 = \frac{x_2 x_3}{x_1}$, $u_2 = \frac{x_3 x_1}{x_2}$ and $u_3 = \frac{x_1 x_2}{x_3}$, then prove that $J(u_1, u_2, u_3) = 4$

Solution: Given that

$$u_1 = \frac{x_2 x_3}{x_1}, \quad u_2 = \frac{x_3 x_1}{x_2} \ \& \ u_3 = \frac{x_1 x_2}{x_3} \quad \text{—— (i)}$$

Partially differentiating (i) w.r.to x_1, x_2 & x_3 respectively, we have

$$\frac{\partial u_1}{\partial x_1} = -\frac{x_2 x_3}{x_1^2}, \quad \frac{\partial u_2}{\partial x_1} = \frac{x_3}{x_2}, \quad \frac{\partial u_3}{\partial x_1} = \frac{x_2}{x_3} \quad \text{—— (ii)}$$

$$\frac{\partial u_1}{\partial x_2} = \frac{x_3}{x_1}, \quad \frac{\partial u_2}{\partial x_2} = -\frac{x_3 x_1}{x_2^2}, \quad \frac{\partial u_3}{\partial x_2} = \frac{x_1}{x_3} \quad \text{—— (iii)}$$

$$\& \ \frac{\partial u_1}{\partial x_3} = \frac{x_2}{x_1}, \quad \frac{\partial u_2}{\partial x_3} = \frac{x_1}{x_2}, \quad \frac{\partial u_3}{\partial x_3} = -\frac{x_1 x_2}{x_3^2} \quad \text{—— (iv)}$$

respectively.

We know that

$$J(u_1, u_2, u_3) = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{vmatrix} = \begin{vmatrix} -\frac{x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & -\frac{x_3 x_1}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & -\frac{x_1 x_2}{x_3^2} \end{vmatrix}$$

(using eqns (ii), (iii), (iv))

$$= \begin{vmatrix} -\frac{x_2 x_3}{x_1^2} & \frac{x_3 x_1}{x_1^2} & \frac{x_2 x_1}{x_1^2} \\ \frac{x_3 x_2}{x_2^2} & -\frac{x_3 x_1}{x_2^2} & \frac{x_1 x_2}{x_2^2} \\ \frac{x_2 x_3}{x_3^2} & \frac{x_1 x_3}{x_3^2} & -\frac{x_1 x_2}{x_3^2} \end{vmatrix}$$

Taking $\frac{1}{x_1^2}, \frac{1}{x_2^2}, \frac{1}{x_3^2}$ common out from rows R_1, R_2, R_3 respectively, we have

$$J(u_1,u_2,u_3) = \begin{vmatrix} -x_2x_3 & x_3x_1 & x_1x_2 \\ x_2x_3 & -x_3x_1 & x_1x_2 \\ x_2x_3 & x_3x_1 & -x_1x_2 \end{vmatrix} \cdot \frac{1}{x_1^2x_2^2x_3^2}$$

Taking x_2x_3, x_3x_1, x_1x_2 common out from colums C_1, C_2, C_3 respectively, we have

$$J(u_1,u_2,u_3) = \frac{(x_2x_3)(x_3x_1)(x_1x_2)}{x_1^2x_2^2x_3^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 2 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \quad (R_1 \to R_1 + R_2)$$

$$= 0 - 0 + 2\cdot(1+1) = 2\times 2 = 4 \quad \left(\begin{array}{l}\text{Expanding} \\ \text{w.r.t. } R_1\end{array}\right)$$

$J(u_1,u_2,u_3) = 4$ (Hence proved)

Exercise :-

(1) If $x = r\cos\theta$, $y = r\sin\theta$, then prove that $J(r,\theta) = \frac{1}{r}$ i.e. prove that $\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r}$.

(2) If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, then find $J(x,y,z)$.

(3) If $y_1 + y_2 + \cdots + y_n = x_1$, $y_2 + y_3 + \cdots + y_n = x_1x_2$, ..., $y_r + y_{r+1} + \cdots + y_n = x_1x_2x_3\cdots x_r$, ..., $y_n = x_1x_2\cdots x_n$, then show that

$$\frac{\partial(y_1,y_2,\ldots,y_n)}{\partial(x_1,x_2,\ldots,x_n)} = x_1^{n-1}x_2^{n-2}\cdots x_{n-2}^2 \cdot x_{n-1}$$